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Temperature dependence of the Casimir effect

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Abstract

In view of the increasing accuracy of Casimir experiments, there is a need for performing accurate theoretical calculations. Using accurate experimental data for the permittivities we present, via the Lifshitz formula applied to the standard Casimir setup with two parallel plates, accurate theoretical results in the case of the metals Au, Cu and Al. Both similar and dissimilar cases are considered. Concentrating in particular on the finite temperature effect, we show how the Casimir pressure varies with separation for three different temperatures, $T = \{1, 300, 350\}$ K. The metal surfaces are taken to be perfectly plane. The experimental data for the permittivities generally yield results that are in a good agreement with those calculated from the Drude relation with finite relaxation frequency. We give the results in a tabular form, in order to facilitate the assessment of the temperature correction which is on the 1% level. We emphasize two points: (i) the most promising route for a definite experimental verification of the finite temperature correction appears to be to concentrate on the case of *large* separations (optimum around $2 \mu\text{m}$); and (ii) there is no conflict between the present kind of theory and the Nernst theorem in thermodynamics.

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1. Introduction

Consider the standard Casimir configuration, namely two semi-infinite homogeneous media separated by a vacuum gap of width a (recent reviews on the Casimir effect are given in [1–4]). Assume that the surfaces are perfectly plane, and that they are of infinite extent. Denote the left-hand slab by 1, the intermediate vacuum region by 2 and the right-hand slab by 3. The two permittivities are $\varepsilon_1(\omega)$ and $\varepsilon_3(\omega)$. Spatial dispersion is neglected. Working in terms of complex frequencies ζ , we have $\omega = i\zeta$. The Lifshitz variables s and p , and the Matsubara

frequencies ζ_m , are

$$s = \sqrt{\varepsilon - 1 + p^2}, \quad p = \frac{q}{\zeta_m}, \quad \zeta_m = \frac{2\pi m}{\beta}, \quad (1)$$

where $\beta = 1/T$ is the inverse temperature, and m is the Matsubara integer (we usually put $\hbar = c = 1$). The nondimensional frequency y and the nondimensional temperature γ are defined by

$$y = qa, \quad q = \sqrt{k_{\perp}^2 + \zeta_m^2}, \quad \gamma = \frac{2\pi a}{\beta}, \quad (2)$$

\mathbf{k}_{\perp} being the transverse wave vector (i.e. parallel to the surfaces). Now defining quantities Δ 's related to the TE and TM modes by

$$\begin{aligned} \Delta_1^{\text{TE}} &= \frac{s_1 - p}{s_1 + p}, & \Delta_2^{\text{TE}} &= \frac{s_3 - p}{s_3 + p}, \\ \Delta_1^{\text{TM}} &= \frac{\varepsilon_1 p - s_1}{\varepsilon_1 p + s_1}, & \Delta_2^{\text{TM}} &= \frac{\varepsilon_3 p - s_3}{\varepsilon_3 p + s_3}, \end{aligned} \quad (3)$$

we can write the Casimir pressure as

$$\mathcal{F} = -\frac{1}{\pi\beta a^3} \sum_{m=0}^{\prime} \int_{m\gamma}^{\infty} y^2 dy \left[\frac{\Delta_1^{\text{TM}} \Delta_2^{\text{TM}} e^{-2y}}{1 - \Delta_1^{\text{TM}} \Delta_2^{\text{TM}} e^{-2y}} + \frac{\Delta_1^{\text{TE}} \Delta_2^{\text{TE}} e^{-2y}}{1 - \Delta_1^{\text{TE}} \Delta_2^{\text{TE}} e^{-2y}} \right], \quad (4)$$

where the prime on the summation sign means that the $m = 0$ term is counted with half weight.

The above formalism summarizes the exposition given recently in [5]. The main purpose of the present paper is the following:

- (1) We calculate the Casimir pressure more accurately than we did earlier, inserting updated permittivity data for the metals Au, Cu and Al (courtesy of Astrid Lambrecht), and extend also the region of separation to larger values. The numerical corrections are small, on the 1% level, but the generally improving accuracy of pressure experiments accentuates the need for working to this degree of accuracy. As before, we take the plates to be perfectly smooth; roughness corrections have to be dealt with separately and are not covered here.

A general property of the Casimir pressure is that from $T = 0$ onwards it *decreases* with increasing values of T , given a fixed value of a . However when the separation is large, $a > 3 \mu\text{m}$, the pressure increases with increasing T when T becomes high. Our numerical results indicate that the pressure increases from around room temperature and upwards. As a general conclusion, a separation of about $2 \mu\text{m}$ appears to be optimal for testing the temperature correction to the Casimir pressure; then the relative correction is highest (cf. also figure 5 in [5]). The practical problem, of course, is that at large separations the Casimir pressure itself is small.
- (2) We next emphasize the point that there is no conflict between this kind of theory (which is equivalent to adopting the Drude dispersion relation) and the Nernst theorem in thermodynamics. According to this theorem, the entropy at zero temperature has to be equal to zero. The Nernst theorem is satisfied in our case, in spite of the fact that the contribution from the TE zero mode (that is, the $m = 0$ term in the Matsubara sum) is equal to zero for finite ε as well as for a real metal. The point here for a real metal is that the relaxation frequency stays different from zero. The first to emphasize this kind of behaviour were Boström and Sernelius [6]. We have treated these thermodynamical issues ourselves also [7–10], and there are several other works expressing the same opinion [11–14]. We mention, though, that the opposite view has also been advocated in recent papers [15–17].

Table 1. The Casimir pressure between Au–Au plates versus gap width a , when $T = \{1, 300, 350\}$ K. The pressures are given in mPa.

a (μm)	$T = 1$ K	$T = 300$ K	$T = 350$ K
0.16	1144	1127	1124
0.2	508.2	497.8	495.7
0.4	38.61	36.70	36.35
0.5	16.56	15.49	15.30
0.7	4.556	4.127	4.052
1.0	1.143	0.9852	0.9590
1.5	0.2342	0.1856	0.1787
2.0	7.549×10^{-2}	5.550×10^{-2}	5.344×10^{-2}
2.5	3.128×10^{-2}	2.176×10^{-2}	2.135×10^{-2}
3.0	1.520×10^{-2}	1.033×10^{-2}	1.049×10^{-2}
3.5	8.252×10^{-3}	5.674×10^{-3}	5.990×10^{-3}
4.0	4.858×10^{-3}	3.481×10^{-3}	3.804×10^{-3}

2. Calculation and results

Let us first recall the Drude dispersion relation

$$\varepsilon(i\zeta) = 1 + \frac{\omega_p^2}{\zeta(\zeta + \nu)}, \tag{5}$$

where ω_p is the plasma frequency and ν is the relaxation frequency. The plasma wavelength is $\lambda_p = 2\pi c/\omega_p$. For the three metals mentioned, the corrected data as compared with those given in [5] are

$$\begin{aligned} \omega_p &= 9.03 \text{ eV}, & \nu &= 34.5 \text{ meV}, & \lambda_p &= 137.4 \text{ nm} & \text{Au}, \\ \omega_p &= 8.97 \text{ eV}, & \nu &= 29.5 \text{ meV}, & \lambda_p &= 138.3 \text{ nm} & \text{Cu}, \\ \omega_p &= 11.5 \text{ eV}, & \nu &= 50.6 \text{ meV}, & \lambda_p &= 107.9 \text{ nm} & \text{Al}. \end{aligned} \tag{6}$$

These corrections are roughly on the 1% level. As before, we calculate the Casimir pressure by means of MATLAB, extracting the zero-frequency case $m = 0$ for a separate analytical treatment. Since ε becomes very large in the zero-frequency limit for metals, we can express the $m = 0$ contribution as

$$\mathcal{F}_0 = \frac{1}{\pi\beta a^3} I_0, \tag{7}$$

where the polylog function with arguments (3,1) is involved,

$$I_0 = -\frac{1}{2} \int_0^\infty y^2 dy \frac{e^{-2y}}{1 - e^{-2y}} = -\frac{1}{8} \text{polylog}(3, 1) = -0.1502571129. \tag{8}$$

The calculated values of the Casimir pressure for Al, Cu and Al are shown in tables 1–6, both for the similar and the dissimilar cases. As in [5], we took the temperature $T = 1$ K to represent the case $T = 0$ case with a good accuracy. In the calculations our tolerance for the integrals was 10^{-12} , whereas the tolerance in the sum was 10^{-8} . At $T = 1$ K the necessary number of terms was quite large, especially at small separations (for instance, about 25 700 at $a = 0.16 \mu\text{m}$).

It is seen that the room-temperature pressure is always weaker than the zero-temperature pressure. Thus for Au–Au plates at separation $a = 0.5 \mu\text{m}$, the pressure is lowered from 16.56 mPa to 15.49 mPa, or by 6.5%. The reduction becomes much enhanced at larger

Table 2. Same as table 1, but for Cu–Cu plates.

a (μm)	$T = 1$ K	$T = 300$ K	$T = 350$ K
0.16	1141	1123	1120
0.2	507.4	496.8	49.47
0.4	38.63	36.69	36.34
0.5	16.57	15.49	15.30
0.7	4.560	4.127	4.052
1.0	1.145	0.9854	0.9592
1.5	0.2345	0.1857	0.1787
2.0	7.559×10^{-2}	5.551×10^{-2}	5.345×10^{-2}
2.5	3.132×10^{-2}	2.177×10^{-2}	2.135×10^{-2}
3.0	1.522×10^{-2}	1.033×10^{-2}	1.049×10^{-2}
3.5	8.263×10^{-3}	5.674×10^{-3}	5.990×10^{-3}
4.0	4.864×10^{-3}	3.481×10^{-3}	3.805×10^{-3}

Table 3. Same as table 1, but for Al–Al plates.

a (μm)	$T = 1$ K	$T = 300$ K	$T = 350$ K
0.16	1290	1271	1267
0.2	565.3	553.9	551.6
0.4	41.17	39.15	38.77
0.5	17.45	16.34	16.13
0.7	4.734	4.290	4.212
1.0	1.175	1.012	0.9853
1.5	0.2383	0.1889	0.1818
2.0	7.648×10^{-2}	5.617×10^{-2}	5.404×10^{-2}
2.5	3.160×10^{-2}	2.195×10^{-2}	2.150×10^{-2}
3.0	1.533×10^{-2}	1.039×10^{-2}	1.053×10^{-2}
3.5	8.311×10^{-3}	5.693×10^{-3}	6.003×10^{-3}
4.0	4.888×10^{-3}	3.488×10^{-3}	3.809×10^{-3}

Table 4. Same as table 1, but for Au–Cu plates.

a (μm)	$T = 1$ K	$T = 300$ K	$T = 350$ K
0.16	1143	1125	1122
0.2	507.8	497.3	495.2
0.4	38.62	36.70	36.34
0.5	16.56	15.49	15.30
0.7	4.558	4.127	4.052
1.0	1.144	0.9853	0.9591
1.5	0.2343	0.1857	0.1787
2.0	7.554×10^{-2}	5.550×10^{-2}	5.345×10^{-2}
2.5	3.130×10^{-2}	2.177×10^{-2}	2.135×10^{-2}
3.0	1.521×10^{-2}	1.033×10^{-2}	1.049×10^{-2}
3.5	8.258×10^{-3}	5.674×10^{-3}	5.990×10^{-3}
4.0	4.861×10^{-3}	3.481×10^{-3}	3.805×10^{-3}

separations; thus at $a = 2 \mu\text{m}$ the pressure is lowered from 7.549×10^{-2} mPa to 5.550×10^{-2} mPa, or by 26.5%. It thus seems advantageous to work with high separations, if technically possible. The differences between Au–Au and Cu–Cu pressures are generally small, whereas the pressures for Al–Al are larger, as we might expect from the dispersive data in (6).

Table 5. Same as table 1, but for Au–Al plates.

a (μm)	$T = 1$ K	$T = 300$ K	$T = 350$ K
0.16	1213	1195	1191
0.2	535.4	524.5	522.3
0.4	39.85	37.89	37.52
0.5	16.99	15.90	15.70
0.7	4.643	4.207	4.130
1.0	1.159	0.9986	0.9720
1.5	0.2362	0.1873	0.1802
2.0	7.598×10^{-2}	5.583×10^{-2}	5.374×10^{-2}
2.5	3.144×10^{-2}	2.185×10^{-2}	2.142×10^{-2}
3.0	1.527×10^{-2}	1.036×10^{-2}	1.051×10^{-2}
3.5	8.281×10^{-3}	5.684×10^{-3}	5.996×10^{-3}
4.0	4.873×10^{-3}	3.485×10^{-3}	3.807×10^{-3}

Table 6. Same as table 1, but for Cu–Al plates.

a (μm)	$T = 1$ K	$T = 300$ K	$T = 350$ K
0.16	1211	1193	1189
0.2	535.0	524.0	521.8
0.4	39.86	37.89	37.52
0.5	17.00	15.90	15.70
0.7	4.646	4.207	4.130
1.0	1.159	0.9987	0.9720
1.5	0.2364	0.1873	0.1802
2.0	7.603×10^{-2}	5.584×10^{-2}	5.375×10^{-2}
2.5	3.146×10^{-2}	2.186×10^{-2}	2.143×10^{-2}
3.0	1.528×10^{-2}	1.036×10^{-2}	1.051×10^{-2}
3.5	8.287×10^{-3}	5.684×10^{-3}	5.997×10^{-3}
4.0	4.876×10^{-3}	3.485×10^{-3}	3.807×10^{-3}

As for the cases where dissimilar metals are involved, the Au–Cu data (table 4) are quite similar to those given in tables 1 and 2. The data for Au–Al and Cu–Al (tables 5 and 6) show somewhat larger pressures.

In the tables we show also the pressures when $T = 350$ K, since one may expect that the pressure difference between 300 K and 350 K will be soon measurable. Again considering Au–Au at $a = 0.5 \mu\text{m}$, we see that the pressure is lowered from 15.49 mPa to 15.30 mPa, or by 1.2%, when T is increased from 300 K to 350 K. If $a = 2 \mu\text{m}$, the corresponding pressure decrease is 3.7%.

A striking property is that for the larger separations, the pressure *increases* with increasing values of T . This turns out numerically when a becomes larger than about $2.8 \mu\text{m}$.

Finally, the following point should be noted: we did not have to use the Drude relation in any of our calculations, for any finite frequency. All the frequencies that were needed, were lying within the region of Lambrecht's data. We needed the Drude relation explicitly at only one place, namely in the evaluation of the zero-frequency term, $m = 0$.

3. On Nernst's theorem

It is important to ensure that the present formalism does not come into conflict with basic thermodynamics. In the present case this means in particular that the Casimir entropy per unit

area at zero temperature,

$$S = - \left(\frac{\partial F}{\partial T} \right)_v, \quad (9)$$

(F being the free energy), has to be zero at $T = 0$. This is Nernst's theorem. Let us make some brief remarks on this topic, in view of its current interest.

- There exist no measurements of the permittivities at very low frequencies. What is at our disposal, is a series of measurements of room-temperature complex frequencies $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$, where the data on $\varepsilon''(\omega)$ permit us to calculate the real quantities $\varepsilon(i\zeta)$ via the Kramers–Kronig relation. The permittivity data received from Lambrecht cover the frequency region 1.5×10^{11} rad s⁻¹ to 1.5×10^{18} rad s⁻¹. Based upon these data, the relaxation frequency ν in the Drude relation is determined. For low frequencies we have to describe the permittivity analytically, with the use of the Drude relation, down to $\zeta = 0$. It is not very important, however, to know the value of ν at $T = 0$ very accurately; the important point is that ν at $T = 0$ stays finite. In practice, this condition is always fulfilled because of scattering from impurities. Then, it is easy to show that the zero-frequency TE mode does not contribute to the Casimir effect. The calculation is shown explicitly in appendix A in [7].
- The above remarks were related to room temperatures. If we proceed to consider low temperatures, we observe that nor in this case there exist permittivity measurements. We thus again have to take recourse to the Drude relation in which, in principle, the value of ν can be different from that above. It seems, however, that the physical significance of an altered value of ν is only minor (a discussion on this point is given in [9]). This is partly due to the impurities, as mentioned above, resulting in scattering also when the temperature is low. Important in the present context is that ν stays finite when $T \rightarrow 0$, so that the behaviour is essentially as above: there is no contribution from the zero-frequency TE mode to the Casimir force. Mathematically, the essential point is that

$$\zeta^2[\varepsilon(i\zeta) - 1] \rightarrow 0 \quad \text{as} \quad \zeta \rightarrow 0. \quad (10)$$

Correspondingly, if the free energy F is drawn as a function of T at some fixed value of a , it turns out numerically, to a high precision, that the slope of the curve is zero at $T = 0$. This result was indicated in figure 5 in [7]; we intend to deal with the topic in more detail in [18]. That is, Nernst's theorem is found to be well satisfied numerically.

- In the above argument the existence of impurities played a certain role, ensuring that ν stays different from zero at all temperatures. Now, it is legitimate to ask: what about the ideal case where the metal is entirely free from impurities? This question, although academic, is nevertheless of fundamental interest.

We may in this context recall the Bloch–Grüneisen formula for the temperature dependence of the electrical resistivity ρ [19]. From this one may estimate the temperature dependence of $\nu = \nu(T)$ to be [9]

$$\nu(T) = 0.0847 \left(\frac{T}{\Theta} \right)^5 \int_0^{\Theta/T} \frac{x^5 e^x dx}{(e^x - 1)^2}, \quad (11)$$

where $\Theta = 175$ K for gold. The Bloch–Grüneisen argument neglects the effect from impurities. It is seen that $\nu(T) \rightarrow 0$ when $T \rightarrow 0$, so that this case becomes indeterminate. To deal with this situation, additional physical effects have to be taken into consideration:

- (1) One way is to include *spatial dispersion*, as was recently done in [13]. One finds by this extension of the theory practically the same results as we did above: there is only a

negligible contribution to the Casimir force from the zero-frequency TE mode. Moreover, the Nernst theorem is found to be satisfied, so that the presence of dissipation or a finite relaxation frequency in the material is not a necessity for this theorem to hold.

- (2) Another approach is to take into account the *anomalous skin effect* [20, 21]. This effect is physically due to the mean free path in the metal being much larger than the field penetration depth near $T = 0$. Again, the results are found to be essentially the same as above: there is no contribution to the Casimir force from the zero TE mode, and there is no contradiction with the Nernst theorem.

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